The role of Large Cardinals in the semantics of Generalized Provability Logics

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Gödel's Provability predicate

Recall that Gödel's provability predicate $\mathit{Prov}(x)$ for Peano's Arithmetic (PA) asserts that x is (a natural number coding) a formula provable from PA.

By Gödel's Second Incompleteness Theorem, if PA is consistent then it does not prove

$$\neg \mathit{Prov}(\ulcorner \bot \urcorner)$$

But PA does prove the following properties of $\mathit{Prov}(x)$:

Distribution

$$\mathit{Prov}(\ulcorner \varphi \to \psi \urcorner) \to (\mathit{Prov}(\ulcorner \varphi \urcorner) \to \mathit{Prov}(\ulcorner \psi \urcorner))$$

Formalized Löb's theorem

$$\mathit{Prov}(\ulcorner \mathit{Prov}(\ulcorner \varphi \urcorner) \to \varphi \urcorner) \to \mathit{Prov}(\ulcorner \varphi \urcorner)$$

$Prov(x)$ as a modality

Distribution

$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

Formalized Löb's theorem

$\Box(\Box\varphi \to \varphi) \to \Box\varphi$

The Gödel-Löb Provability Logic (GL)

GL is a logic in the language of propositional logic with an additional modal operator $\Box$.

Axioms:

1. Propositional tautologies.
2. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ (Distribution)
3. $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ (GL axiom)

Rules:

1. $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$ (Modus Ponens)
2. $\vdash \varphi \Rightarrow \vdash \Box\varphi$ (Necessitation)

Completeness

Theorem (Segerberg 1971)

GL *is sound and complete under Kripke semantics for the class of finite irreflexive trees.*

This is a key point in the remarkable proof by Solovay that GL is arithmetically complete, i.e., GL completely axiomatizes the properties of $\mathrm{Prov}(x)$ that are provable in PA.

Solovay's Theorem

Theorem (Solovay 1976)

$\mathrm{GL} \vdash \varphi$ *if and only if* $\mathrm{PA} \vdash f(\varphi)$ *for all realizations* f.

A realization f is a function from modal to arithmetical formulas that assigns an arithmetical sentence to each propositional variable, it respects all connectives, and $f(\Box\varphi) = Prov(\ulcorner f(\varphi) \urcorner)$.

Polymodal provability logics

Japaridze (1986) considers two modalities: $[0]$, corresponding to the $Prov$ predicate, and $[1]$ corresponding to the $Prov_1$ predicate, i.e., provability in PA together with all Π_1 truths.

The corresponding logic, called GLB, has the same axioms as GL for each of the modalities, plus two mixed axioms:

$$[0]\varphi \to [1]\varphi \qquad \text{(Monotonicity)}$$

$$\langle 0\rangle\varphi \to [1]\langle 0\rangle\varphi \qquad (\Pi_1\text{-completeness})$$

Here, $\langle 0\rangle$ is the dual operator $\neg[0]\neg$.

The Logic GLP_ω (Japaridze, 1986)

He also considers the language of propositional logic with additional modal operators $[n]$, for each $n < \omega$. The logic system GLP_ω has the following axioms and rules:

Axioms:

1. Propositional tautologies.
2. $[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$, for all $n < \omega$
3. $[n]([n]\varphi \to \varphi) \to [n]\varphi$, for all $n < \omega$
4. $[m]\varphi \to [n]\varphi$, for all $m < n < \omega$
5. $\langle m\rangle\varphi \to [n]\langle m\rangle\varphi$, for all $m < n < \omega$

Rules:

1. $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$ (Modus Ponens)
2. $\vdash \varphi \Rightarrow \vdash [n]\varphi$, for all $n < \omega$ (Necessitation)

The Logic GLP_ξ

More generally, for any ordinal $\xi \geqslant 2$, one considers the language of propositional logic with additional modal operators $[\alpha]$, for each $\alpha < \xi$. The logic system GLP_ξ has the following axioms and rules:

Axioms:

1. Propositional tautologies.
2. $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$, for all $\alpha < \xi$
3. $[\alpha]([\alpha]\varphi \to \varphi) \to [\alpha]\varphi$, for all $\alpha < \xi$
4. $[\beta]\varphi \to [\alpha]\varphi$, for all $\beta < \alpha < \xi$
5. $\langle\beta\rangle\varphi \to [\alpha]\langle\beta\rangle\varphi$, for all $\beta < \alpha < \xi$

Rules:

1. $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$ (Modus Ponens)
2. $\vdash \varphi \Rightarrow \vdash [\alpha]\varphi$, for all $\alpha < \xi$ (Necessitation)

Topological semantics (Simmons, Esakia, 1970s)

Problem

The logic GLB *(and hence also* GLP_ξ *for* $\xi \geqslant 2$*) is not complete with respect to any class of Kripke frames.*

Thus, one turns to topological semantics, where $\langle 0\rangle$ is interpreted as the Cantor derivative operator, i.e.,

$$v(\langle 0\rangle\varphi) = \mathrm{d}(v(\varphi)) = \{x : x \text{ is a limit point of } v(\varphi)\}.$$

Topologial semantics

We consider polytopological spaces $(X, (\tau_\alpha)_{\alpha<\xi})$.

A valuation on X is a map v that assigns to each formula of GLP_ξ a subset of X so that:

1. $v(\neg\varphi) = X - v(\varphi)$
2. $v(\varphi \wedge \psi) = v(\varphi) \cap v(\psi)$
3. $v(\langle n \rangle \varphi) = d_\alpha(v(\varphi))$, for all $\alpha < \xi$, where

   $d_\alpha : \mathcal{P}(X) \to \mathcal{P}(X)$ is the derived set operator for τ_α (i.e., $d_\alpha(A)$ is the set of limit points of A in the τ_α topology). Hence, $v([\alpha]\varphi) = X - d_\alpha(X - v(\varphi))$, for all $\alpha < \xi$.

A formula is valid in $(X, (\tau_\alpha)_{\alpha<\xi})$ if $v(\varphi) = X$, for every valuation v on X.

GLP spaces

A space $(X, (\tau_\alpha)_{\alpha<\xi})$ is called a GLP-space if:

1. (X, τ_0) is scattered
2. $\tau_\alpha \subseteq \tau_{\alpha+1}$
3. $d_\alpha(A)$ is an open set in $\tau_{\alpha+1}$, for all $A \subseteq X$.

Theorem (Beklemishev-Gabelaia[1])

GLP_ω *is complete with respect to the class of GLP-spaces.*

Theorem (Fernández-Duque[2])

If ξ *is countable, then* GLP_ξ *is complete with respect to the class of GLP-spaces.*

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[1] *Topological completeness of the provability logic GLP*, Annals of Pure and Applied Logic 164 (12), 1201-1223 (2013)

[2] *The polytopologies of transfinite provability logic*. Archive for Mathematical Logic, Volume 53, Issue 3, 385-431 (2014)

Ordinal topologial semantics

The most natural class of scattered spaces are ordinal numbers with the usual interval topology. So, the question is:

Question

Is GLP_ξ (for $\xi \geqslant 2$) complete with respect to ordinal GLP spaces?

We consider GLP spaces $(\Omega, (\tau_\alpha)_{\alpha<\xi})$, where Ω is some limit ordinal, or OR, and where

1. τ_0 is the interval topology.
2. $\tau_{\alpha+1}$ is the topology generated by $\tau_\alpha \cup \{d_\alpha(A) : A \subseteq \Omega\}$, where

   $d_\alpha(A) := \{\gamma < \Omega : \gamma \text{ is a limit point of } A \text{ in the } \tau_\alpha \text{ topology}\}.$
3. $\tau_\alpha = \bigcup_{\beta<\alpha} \tau_\beta$, if α is a limit ordinal.

Ordinal topological semantics

Note that if the cofinality of β is uncountable and $\beta \in d_0(A)$, then $d_0(A) \cap \beta$ is a club subset of β.

Since the set $\tau_0 \cup \{d_0(A) : A \subseteq \Omega\}$ generates the topology τ_1 on Ω, τ_1 is known as the club topology.

The non-isolated points of τ_1 are exactly the ordinals of uncountable cofinality.

Theorem (Soundness)

Every GLP_ξ-provable formula is valid in $(\Omega, (\tau_\alpha)_{\alpha<\xi})$.

Completeness of GLP_2

Theorem (Blass 1990. Beklemishev 2011)

If $\square_\kappa$ holds for all $\kappa < \aleph_\omega$, then GLP_2 is complete with respect to any $(\Omega, (\tau_0, \tau_1))$, with $\Omega \geqslant \aleph_\omega$. [3]

Theorem (Blass 1990)

It is consistent with ZFC, modulo the existence of a Mahlo cardinal, that GLP_2 is not complete with respect to $(\Omega, (\tau_0, \tau_1))$, for any ordinal Ω.

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[3] *Infinitary Combinatorics and Modal Logic*. The Journal of Symbolic Logic, Vol. 55, No. 2, 761-778 (1990).
Ordinal Completeness of Bimodal Provability Logic GLB. In Logic, Language, and Computation, Lecture Notes in Computer Science, Vol. 6618, 1-15 (2011).

On discreteness

Question

Is it consistent with ZFC that GLP_3 *is complete with respect to some* $(\Omega, (\tau_0, \tau_1, \tau_2))$*?*

Fact

For GLP_ξ *to be complete with respect to some* $(\Omega, (\tau_\alpha)_{\alpha<\xi})$ *we need that all the* τ_α *are non-discrete. (If* τ_α *is discrete, then the non-provable formula* $[\alpha]\perp$ *is valid.)*

Non-isolated points in the τ_2 topology

For every set of ordinals A,

$$d_1(A) = \{\alpha : A \cap \alpha \text{ is stationary in } \alpha\}.$$

If some stationary subset S of α does not reflect (i.e., $d_1(S) = \{\alpha\}$), then α is an isolated point of τ_2. Thus, every non-isolated point α has to reflect stationary sets.

Let us say that an ordinal α is 2-stationary if all stationary subsets of α reflect. It is well-known that the first 2-stationary cardinal, if it exists, must be either weakly inaccessible or the successor of a singular cardinal.

So all ordinals $\alpha \leqslant \aleph_\omega$ are isolated in the τ_2 topology.

But being 2-stationary is not enough.

Stationary reflection

Let us say that a set A of ordinals is 2-simultaneously-stationary, or 2-s-stationary for short, in α if every pair S, T of stationary subsets of α simultaneously reflect on some $\beta \in A$, that is, there exists $\beta < \alpha$ such that $S \cap \beta$ and $T \cap \beta$ are both stationary in β.

This property characterizes the non-isolated points in the τ_2 topology, i.e.,

α is not an isolated point of τ_2 iff α is 2-s-stationary in α.

Theorem (Jensen)

In the constructible universe L a regular cardinal κ is 2-stationary if and only if it is 2-s-stationary, if and only if it is weakly compact.[4]

[4] R. Jensen, The fine structure of the constructible hierarchy. Annals of Math. Logic 4 (1972)

α-stationary sets

In order to characterize the non-isolated points of the τ_α topology, for $\alpha > 2$, we need a generalized notion of stationary reflection.

Definition

We say that $A \subseteq OR$ is 0-stationary in α if and only if $A \cap \alpha$ is unbounded in α.

For $\xi > 0$, we say that A is ξ-stationary in α if and only if for every $\zeta < \xi$, every ζ-stationary subset S of α ζ-reflects to some $\beta \in A$, i.e., $S \cap \beta$ is ζ-stationary in β.

Definition

We say that $A \subseteq OR$ is 0-s-stationary in α if and only if $A \cap \alpha$ is unbounded in α.

For $\xi > 0$, we say that A is ξ-s-stationary in α if and only if for every $\zeta < \xi$, every pair S, T of ζ-s-stationary subsets of α ζ-s-reflect to some $\beta \in A$, i.e., $S \cap \beta$ and $T \cap \beta$ are ζ-s-stationary in β.

Fact

For every ξ,

$$d_\xi(A) = \{\alpha : A \cap \alpha \text{ is } \xi\text{-s-stationary in } \alpha\}.$$

Theorem

For every ξ, an ordinal α is non-isolated in the τ_ξ topology if and only if α is ξ-s-stationary.[5]

[5] In the case $\xi < \omega$ this is due independently to L. Beklemishev.

Reflection and indescribability in L

Recall that a cardinal κ is Π^1_n-indescribable if for every $A \subseteq V_\kappa$ and every Π^1_n-sentence $\varphi(A)$, if

$$\langle V_\kappa, \in, A\rangle \models \varphi(A)$$

then there is $\lambda < \kappa$ such that

$$\langle V_\lambda, \in, A \cap V_\lambda\rangle \models \varphi(A \cap V_\lambda).$$

Theorem (Bagaria-Magidor-Sakai 2015[6])

Assume $V = L$. For every $n > 0$, a regular cardinal is $(n+1)$-stationary if and only if it is $(n+1)$-s-stationary if and only if it is Π^1_n-indescribable.

Thus, in L the non-isolated points of the τ_{n+1} topology are precisely the ordinals whose cofinality is a Π^1_n-indescribable cardinal.

[6] *Reflection and Indescribability in the constructible universe.* Israel Journal of Mathematics. Volume 208, Issue 1, pp 1-11 (2015).

Definition

For ξ any ordinal, a formula is $\Sigma^1_{\xi+1}$ if it is of the form

$$\exists X_0, \ldots, X_k \varphi(X_0, \ldots, X_k)$$

where $\varphi(X_0, \ldots, X_k)$ is Π^1_ξ.
And a formula is $\Pi^1_{\xi+1}$ if it is of the form

$$\forall X_0, \ldots, X_k \varphi(X_0, \ldots, X_k)$$

where $\varphi(X_0, \ldots, X_k)$ is Σ^1_ξ.
If ξ is a limit ordinal, a formula is Σ^1_ξ (respectively Π^1_ξ) if it is of the form

$$\bigvee_{\zeta<\xi} \varphi_\zeta \quad (\text{respectively } \bigwedge_{\zeta<\xi} \varphi_\zeta)$$

where φ_ζ is Σ^1_ζ (respectively Π^1_ζ), all $\zeta < \xi$, and has only finitely-many free second-order variables.

Reflection and indescribability in L

A cardinal κ is Π^1_ξ-indescribable if for every $A \subseteq V_\kappa$ and every Π^1_ξ-sentence $\varphi(A)$, if

$$\langle V_\kappa, \in, A\rangle \models \varphi(A)$$

then there is $\lambda < \kappa$ such that

$$\langle V_\lambda, \in, A \cap V_\lambda\rangle \models \varphi(A \cap V_\lambda).$$

Theorem (Bagaria 2016[7])

Assume $V = L$. For every $\xi > 0$, a regular cardinal is $(\xi + 1)$-stationary if and only if it is $(\xi + 1)$-s-stationary if and only if it is Π^1_ξ-indescribable.

Thus, in L the non-isolated points of $\tau_{\xi+1}$ are precisely the ordinals whose cofinality is a Π^1_ξ-indescribable cardinal.

[7] *Derived topologies on ordinals and stationary reflection*. To appear in TAMS.

Question

What is the consistency strength of ξ-stationarity?
And of ξ-s-stationarity?

The ideal of non-ξ-s-stationary sets

For each limit ordinal α and each $\xi > 0$, let $\mathrm{NS}^{\xi}_{\alpha}$ be the set of non-ξ-s-stationary subsets of α.

Thus, if α has uncountable cofinality, NS^{1}_{α} is the ideal of non-stationary subsets of α and $(\mathrm{NS}^{1}_{\alpha})^*$ is the club filter over α.

Theorem

For every ξ, an ordinal α is ξ-s-stationary if and only if $\mathrm{NS}^{\xi}_{\alpha}$ is a proper ideal.

Fact

If κ is a Π^1_{ξ}-indescribable cardinal, then the ideal $\mathrm{NS}^{\xi+1}_{\kappa}$ is proper, normal, and κ-complete.

On the consistency strength of ξ-stationarity

Definition (Mekler-Shelah 1989)

A regular uncountable cardinal κ is a reflection cardinal if there exists a proper, normal, and κ-complete ideal $\mathcal{I}$ on κ such that for every $X \subseteq \kappa$,

$$X \in \mathcal{I}^+ \quad \Rightarrow \quad d_1(X) \in \mathcal{I}^+.$$

Note: if κ is 2-s-stationary, then NS^1_κ is the smallest such ideal. If κ is weakly compact, then there are many reflection cardinals below κ.

Theorem (Mekler-Shelah 1989)

The following are equiconsistent:

1. *There exists a 2-stationary cardinal.*
2. *There exists a reflection cardinal.*

ξ-reflection cardinals

For a set $X \subseteq \kappa$, let

$$d^*_\xi(X) := \{\alpha < \kappa : \alpha \cap X \text{ is } \xi\text{-stationary in } \alpha\}$$

Definition

A regular uncountable cardinal κ is a ξ-reflection cardinal if there exists a proper, normal, and κ-complete ideal $\mathcal{I}$ on κ such that $d^*_\xi(\kappa)$ belongs to $\mathcal{I}^*$, and for every $X \subseteq \kappa$,

$$X \in \mathcal{I}^+ \quad \Rightarrow \quad d^*_\xi(X) \in \mathcal{I}^+$$

On the consistency strength of ξ-stationarity

Theorem (Bagaria-Magidor-Mancilla, 2017)

Assume $V = L$ and suppose that κ is an n-reflection cardinal. Then, in some forcing extension that preserves cardinals, κ is $(n+1)$-stationary.

If κ is Π^1_n-indescribable, then κ is an n-reflection cardinal, and the set of n-reflection cardinals smaller than κ is n-stationary.
Thus, the consistency strength of an $(n+1)$-stationary cardinal is strictly weaker than the existence of a Π^1_n-indescribable cardinal.

Magidor[8] shows that the following are equiconsistent:

1. There exists a 2-s-stationary cardinal.
2. There exists a weakly-compact (i.e., Π^1_1-indescribable) cardinal.

Question

What is the consistency strength of the existence of an n-s-stationary cardinal? Is it a Π^1_{n-1}-indescribable cardinal?

[8] M. Magidor, On reflecting stationary sets. JSL 47 (1982)